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Comment on Geometry of Hyperspin Manifolds

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Riemannian geometry plays an exceptional role among geometries represented by symmetric tensor $g_{\mu_1 \dots \mu_n}$ of rank n. In particular, Holm's claim that there exist torsion-free Christoffel coefficients on every hyperspin manifold is false.

Recently Finkelstein (1986) has proposed a new idea in physics—a hyperspin theory. The basic dynamical variable in the theory is the spin form, which is an $SL(n, \mathbb{C})$-type object. The spin form also defines the chronometric. While the Riemann geometry is based on a second-rank symmetric tensor $g_{\mu\nu}$, the chronometric is an n-ary ($n \geq 3$) symmetric form $g_{\mu_1 \dots \mu_n}$. A manifold M equipped with such a tensor will be called a hypermanifold. (The name hyperspin manifold is reserved for the case where the chronometric comes from spin form.) Holm (1986) discussed a geometry of hyperspin manifolds in complete analogy to the metric geometry. The purpose of this paper is to explain why some important concepts of Riemannian geometry work differently in the chronometric case and, in particular, to point out some oversimplifications of Holm. Discussion is confined to the case of $n = 3$ (a generalization from 3 to n is evident). Let us begin with a simple algebraic consideration.

Let $\eta = (\eta_{ijk})$ be a symmetric third-rank covariant tensor on a linear space $V \simeq \mathbb{R}^N$, i.e., $\eta \in V^* \odot V^* \odot V^*$, where $\odot$ denotes a symmetric tensor product. Assume that η is nondegenerate in the following sense: the mapping (also denoting by η)

$$V \ni (x^i) \xrightarrow{\eta} (v_{kl} = x^i \eta_{ikl}) \in V^* \odot V^* \tag{1}$$

is injective (I use the Einstein summation convention). The Riemannian case is an exception because (1) becomes an *isomorphism* of V onto V^*.

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Decomposing $V^* \odot V^* = L'' \oplus L'$ into a direct sum, where $L'' = \mathrm{Im}\ \eta$, one can define (depending on the choice of L') the linear mapping called an inverse (or dual) chronometric $\tilde{\eta}$: $V^* \odot V^* \to V$ by

$$\tilde{\eta}_{|L'} = 0; \qquad \tilde{\eta}_{|L''} = \eta^{-1} \tag{2}$$

By the definition, $\tilde{\eta} = (\eta^{ijk})$ is any solution of the equation

$$\eta^{kij} \eta_{ijm} = \delta^k_m \tag{3}$$

Notice that in general η^{kij} will be symmetric only in the last two indices. Even if there exists a completely symmetric solution, it is nonunique. In the following we will fix some dual tensor $\tilde{\eta}$.

It is convenient to introduce the tensor $\lambda^{ij}_{kl} = \eta^{mij} \eta_{mkl}$. Then (3) guarantees the following properties of λ:

$$\begin{gathered} \lambda: \quad V^* \odot V^* \to V^* \odot V^* \\ \lambda^{ij}_{kl} \lambda^{kl}_{mn} = \lambda^{ij}_{mn} \\ \lambda^{ij}_{kl} \eta^{mkl} = \eta^{mij}; \qquad \lambda^{ij}_{kl} \eta_{mij} = \eta_{mkl} \end{gathered} \tag{4}$$

This means that λ is a projection operator on a subspace L'' along the direction L'. Immediately one obtains that the condition

$$\lambda^{ij}_{kl} v_{ij} = v_{kl} \tag{5}$$

is sufficient and necessary for an element $(v_{ij}) \in V^* \odot V^*$ to be a value of some vector $(v^m = \eta^{mij} v_{ij}) \in V$ under the action of (1). Now we proceed to the study of hypermanifolds.

Let M be an N-dimensional manifold with given $G(\eta)$ structure, where $G(\eta)$ is an η-preserving subgroup of $GL(N, \mathbb{R})$. Let $P(G(\eta), M)$ denote the corresponding subbundle of the bundle of linear frames. For every $p \in M$ there exists an open set V and local section

$$V \to x \to e(x) = \{e_a(x)\} \in P(G(\eta), M)\} \tag{6}$$

of η-frames $e_a(x) = e^\mu_a \partial_\mu$. We call e_a an N-bain. $P(G(\eta), M)$ is a g-structure on M (Kobayashi, 1972), where $g = (g_{\mu\nu\lambda})$,

$$g_{\mu\nu\lambda} = e^i_\mu e^j_\nu e^k_\lambda \eta_{ijk} \tag{7}$$

is a global tensor field on M with η being its canonical form. The resulting structure may be called a hypermanifold. In an analogy to the Riemannian case, one would also like to introduce a tensor field $\tilde{g} = (g^{\mu\nu\lambda})$ and $h = (h^{\mu\nu}_{\lambda\rho})$ by

$$\begin{gathered} g^{\mu\nu\lambda} = e^\mu_i e^\nu_j e^\lambda_k \eta^{ijk} \\ h^{\mu\nu}_{\lambda\rho} = g^{\alpha\mu\nu} g_{\alpha\lambda\rho} \end{gathered} \tag{8}$$

where e_i^{μ} is an inverse matrix to e_{μ}^{i}. However, for a nontrivial $P(G(\eta), M)$, only $g_{\mu\nu\lambda}$ is a canonical global tensor field on M. This is because the group $G(\tilde{\eta})$ is in general quite different from $G(\eta)$. Of course, for paracompact M, $\tilde{g}$ and h can be chosen globally, but in many inequivalent ways. In the following let us assume that some fields $\tilde{g}$ and h are fixed on M.

Let $\Gamma_{\mu\nu}^{\lambda}$ be the connection coefficients for some g-connection on M. From $\nabla g = 0$ one gets

$$\partial_{\mu} g_{\nu\lambda\rho} = \Gamma_{\mu\nu\lambda\rho} + \Gamma_{\mu\lambda\rho\nu} + \Gamma_{\mu\rho\nu\lambda} \tag{9}$$

where $\Gamma_{\mu\nu\lambda\rho} = \Gamma_{\mu\nu}^{\kappa} g_{\kappa\lambda\rho}$. Holm (1986) has found a "unique torsion-free" solution of (9) in the form

$$\Omega_{\mu\nu\lambda\rho} = \tfrac{1}{2}(\partial_{\mu} g_{\nu\lambda\rho} + \partial \nu g_{\mu\lambda\rho}) - \tfrac{1}{6}(\partial_{\rho} g_{\lambda\mu\nu} + \partial_{\lambda} g_{\rho\mu\nu}) \tag{10}$$

Indeed, (10) is a solution of (9) and is symmetric in the first pair of indices. Unfortunately, $\Omega_{\mu\nu\lambda\rho}$ do not transform into themselves under gauge transformations, so they depend on coordinates. In particular, (10) *does not yield a global object*. The next mistake is more serious—the solutions (10) are *not connection coefficients at all*. To see this, let us observe that $\Omega_{\mu\nu\alpha\beta}$ does not satisfy the constraint (5),

$$h_{\alpha\beta}^{\lambda\rho} \Omega_{\mu\nu\lambda\rho} = \Omega_{\mu\nu\alpha\beta} \tag{11}$$

hence there do not exist connection coefficients $\Gamma_{\mu\lambda}^{\lambda}$ such that $\Omega_{\mu\nu\alpha\beta} = \Gamma_{\mu\nu}^{\lambda} g_{\lambda\alpha\beta}$.

It can be seen that no algebraic solution of (10) satisfies (11). As an example of a nonalgebraic solution, one can take, e.g., pure gauge, namely $\Omega_{\mu\nu\alpha\beta} = e_i^{\lambda}(\partial_{\mu} e_{\nu}^{i}) g_{\lambda\alpha\beta}$.

For a paracompact M, a g-connection always exists, but the problem of finding a torsion-free (or a Christoffel-like) connection on M remains open.

Therefore one cannot use the geodesic principle of general relativity. Instead, one can use the method of Souriau (1974) [developed by Jadczyk (1983)] in order to obtain particle trajectories in a space-time M with geometry represented by the tensor field $g_{\mu\nu\lambda}$.

Let $\delta g = g' - g$ be a symmetric tensor of type (3, 0). The "golden rule" for matter regularly concentrated on a curve $K \subset M$ can be written in the form

$$\int_{k} (\sigma^{\mu\nu\lambda} \delta g_{\mu\nu\lambda})\, dt = 0 \tag{12}$$

where x^{μ} and t are coordinate systems on M and K, respectively, and $\sigma^{\mu\nu\lambda}$ denotes a density on K with values in symmetric tensor of type (0, 3); $\delta g_{\mu\nu\lambda}$ is of the form $L_Y g_{\mu\nu\lambda}$, with Y being any vector fields on M. Then, after a lengthy calculation and using methods developed by Jadczyk (1983), one finds

$$\frac{d}{t}(v^{\mu}v^{\nu}g_{\mu\nu\eta})-\tfrac{1}{2}v^{\mu}v^{\nu}v^{\rho}\partial_{\lambda}g_{\mu\nu\rho}=0 \tag{13}$$

where $v^{\mu}=dx^{\mu}/dt$ are velocity components. It is remarkable that the same equations can be derived from the variational principle for the functional $\int_a^b (g_{\mu\nu\lambda}v^{\mu}v^{\nu}v^{\lambda})^{1/3}\,dt$.

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